Operations on Maps

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Operations on Maps

- A map *M* is a combinatorial representation of a closed surface.[1]
- Several operations on a map allow its transformation into new maps (convex polyhedra).
- **Platonic** polyhedra: **Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron**

1. Pisanski, T.; Randić, M. Bridges between Geometry and Graph Theory. In: *Geometry at Work*, M. A. A. Notes, 2000, *53*, 174-194.

Euler Theorem on Polyhedra

Any map *M* and its transforms by map operations will obey the Euler theorem[1]

$$v - e + f = \chi = 2(1 - g)$$

χ = *Euler's* characteristic
v = number of vertices,
e = number of edges,
f = number of faces,
g = *genus* ; (g = 0 for a sphere; 1 for a torus).

1. L. Euler, Elementa doctrinae solidorum, *Novi Comment. Acad. Sci. I. Petropolitanae* **1758**, *4*, 109-140.

1. *Dual*

- ***Dualization Du*** of a map: put a point in the center of each face of *M*. Join two points if their corresponding faces share a common edge. The transformed map is called the (Poincaré) *dual* **Du (*M*).**

- The vertices of *Du* (*M*) represent faces in *M* and *vice-versa*. The following relations exist:
- ***Du* (*M*):** $v = f_0$

  $e = e_0$

  $f = v_0$
- Dual of the dual recovers the map itself:

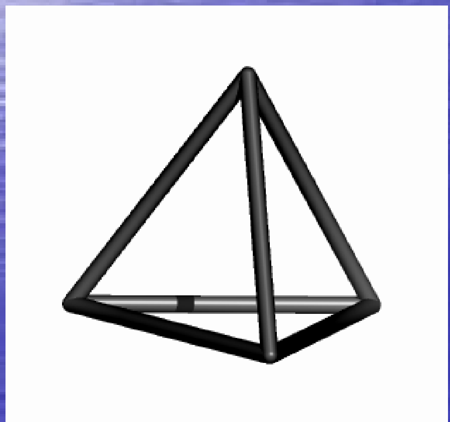
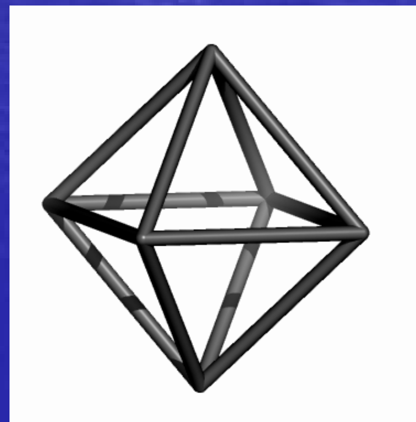
  $Du(Du(M)) = M$

Dual, examples

Du(Tetrahedron) = Tetrahedron

Du(Cube) = Octahedron

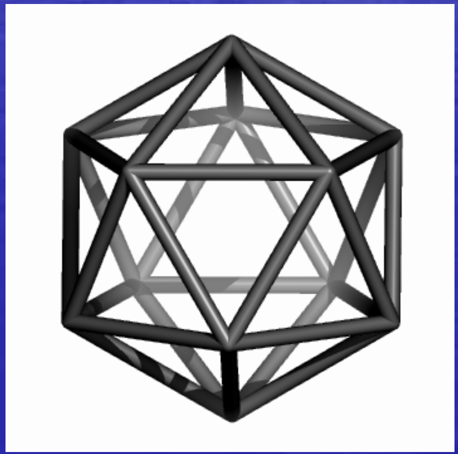
Platonic Solids

Tetrahedron	Cube	Octahedron
		

Dual, examples

Du(Dodecahedron) = Icosahedron

Platonic Solids

Dodecahedron	Icosahedron
	

Dual of a triangulation is always a cubic net.

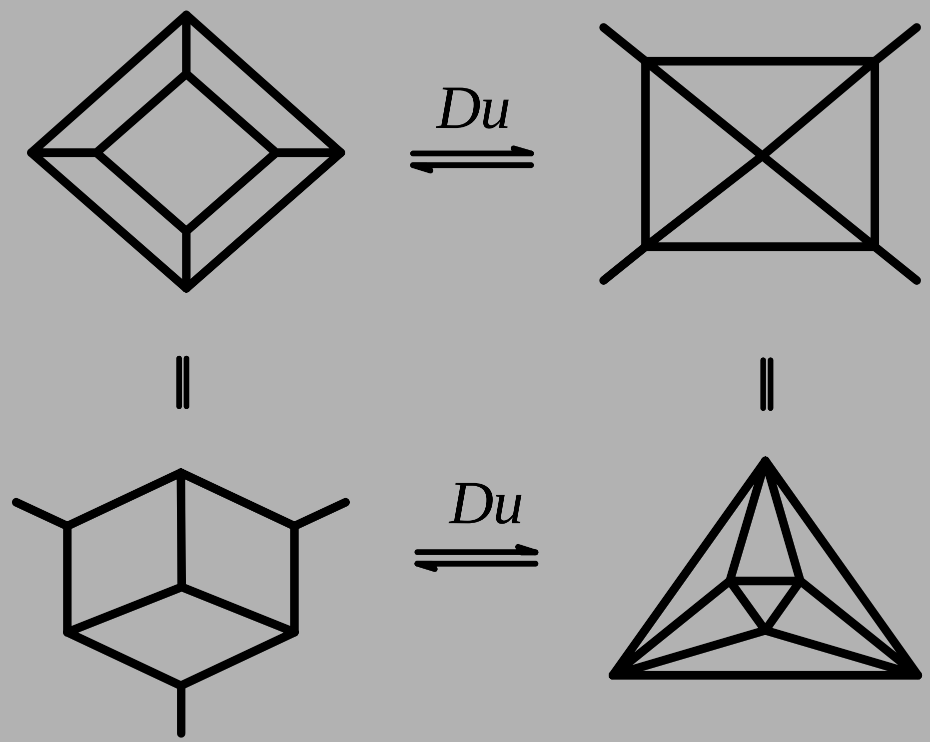
Schlegel projection

- A projection of a sphere-like polyhedron on a plane is called a Schlegel diagram.
- In a polyhedron, the center of diagram is taken either a vertex, the center of an edge or the center of a face

Cube and its dual, Octahedron

Schlegel diagrams

Cube



Octahedron

2. *Medial*

- ***Medial Me*** is achieved as follows:

The new vertices are the midpoints of the original edges. Join two vertices if the original edges span an angle (and are consecutive).

- $Me(M)$ is a 4-valent graph
- $Me(M) = Me(Du(M))$.

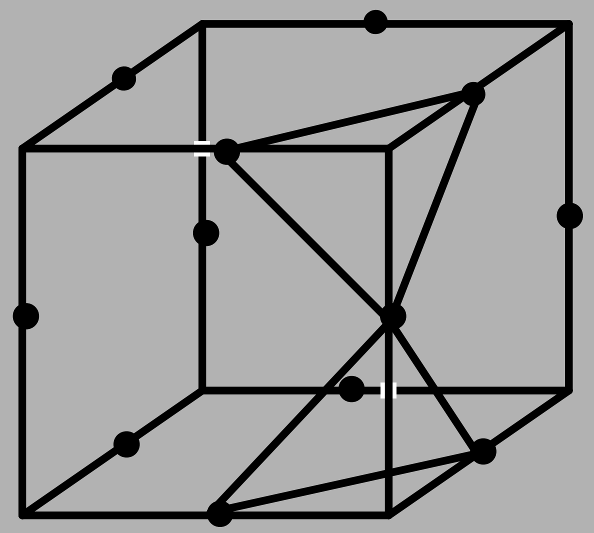
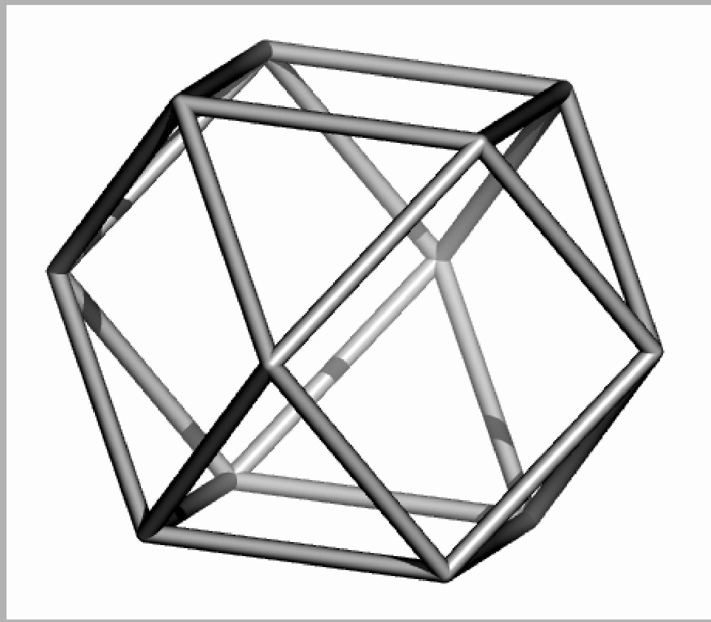
The transformed parameters are:

- ***Me*** **(*****M*****):**

$$v = e_0$$
$$e = 2e_0$$
$$f = f_0 + v_0$$

- ***Me*** operation rotates parent s-gonal faces by π/s.

Medial; example

Subdivided *Su* 1(*C*)	Cubeoctahedron = *Me* (*C*)
	

3. *Truncation*

- **Truncation** ***Tr*** is acheved by cutting of the neighborhood of each vertex by a plane close to the vertex, such that it intersects each edge incident to the vertex.
- Truncation is related to the medial, with the main difference that each old edge will generate three new edges in the truncated map. The transformed parameters are:
- ***Tr*(*M*)**:

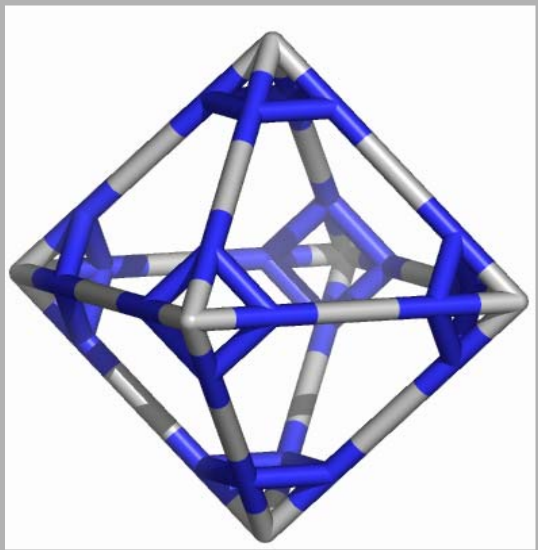
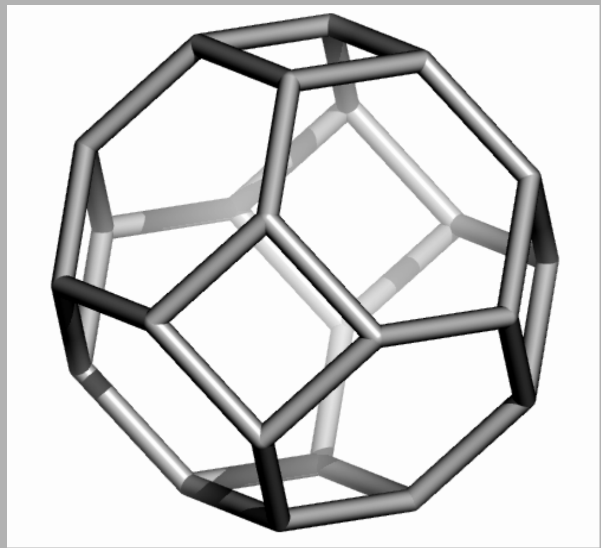
$$v = d_0 v_0$$

$$e = 3e_0$$

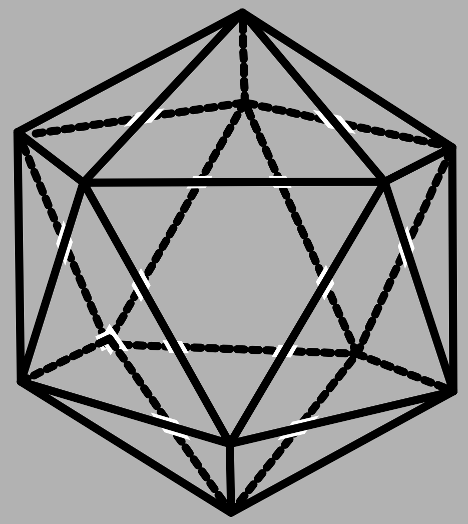
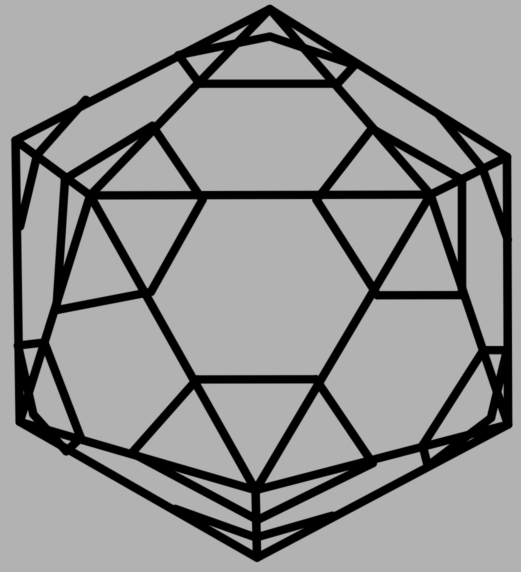
$$f = f_0 + v_0$$

Truncation, example

Tr (*M*) always generates a 3-valent map.

Tr (*Octahedron*)	3D Truncated Octahedron
	

Truncation, example

Icosahedron	*Tr* (*Icosahedron*) = C_{60}
	

4. *Polygonal P_s Capping*

- ***Capping P_s*** (s = 3, 4, 5) of a face is achieved as follows:[1]
- Add a new vertex in the center of the face. Put *s* -3 points on the boundary edges.
- Connect the central point with one vertex (the end points included) on each edge.
- The parent face is thus covered by: trigons (*s* = 3), tetragons (*s* = 4) pentagons (*s* = 5).
- The P_3 operation is also called *stellation* or (centered) *triangulation*.
- When all the faces of a map are thus operated, it is referred to as an omnicapping P_s operation.

1. M. V. Diudea, Covering forms in nanostructures, *Forma* 2004 (submitted)

Polygonal P_s Capping

The resulting map shows the relations:

$P_s(M)$:
$$v = v_0 + (s-3)e_0 + f_0$$
$$e = se_0 \quad ; \; e = s_0 f_0 + (s-2)e_0 = se_0$$
$$f = s_0 f_0$$

Maps transformed as above form *dual pairs* :

$$Du(P_3(M)) = Le(M)$$
$$Du(P_4(M)) = Me(Me(M))$$
$$Du(P_5(M)) = Sn(M)$$

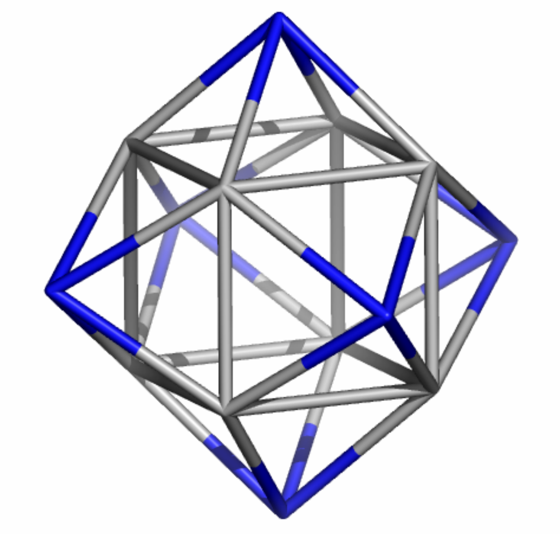
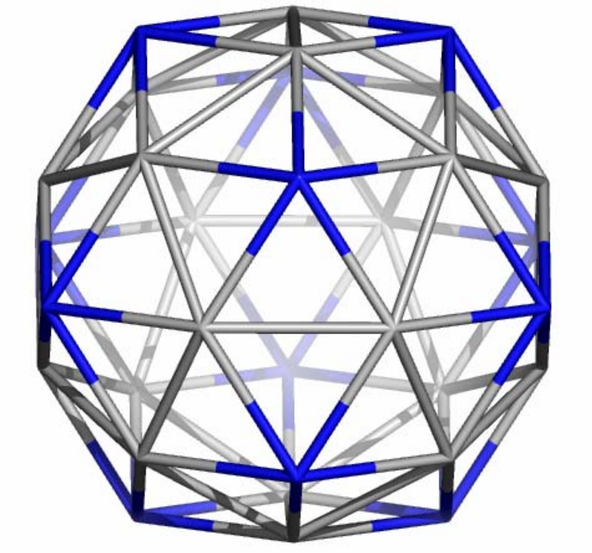
Vertex multiplication ratio by this dualization is always:

$$v(Du)/v_0 = d_0$$

Since:

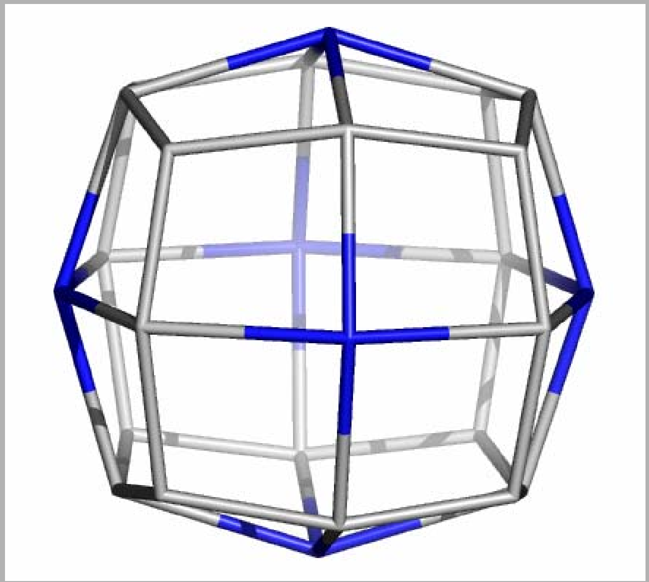
$$v(Du) = f(P_s(M)) = s_0 f_0 = d_0 v_0$$

P_3 Capping = Triangulation

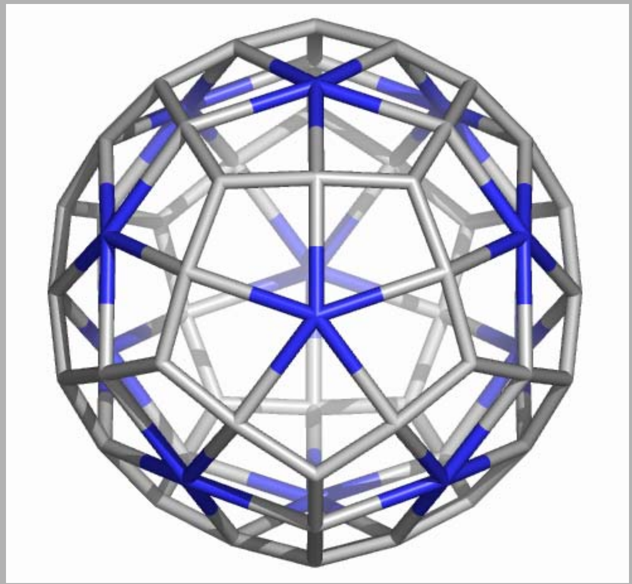
$P_3(Cube)$	$P_3(Dodecahedron)$
	

P_4 Capping = Tetrangulation

$P_4(Cube)^1$

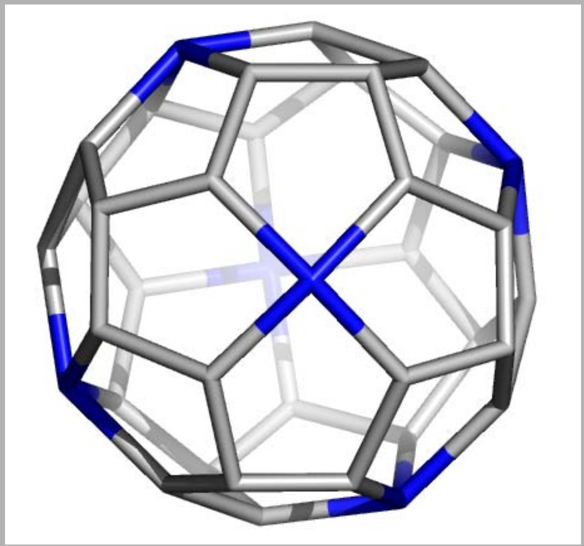
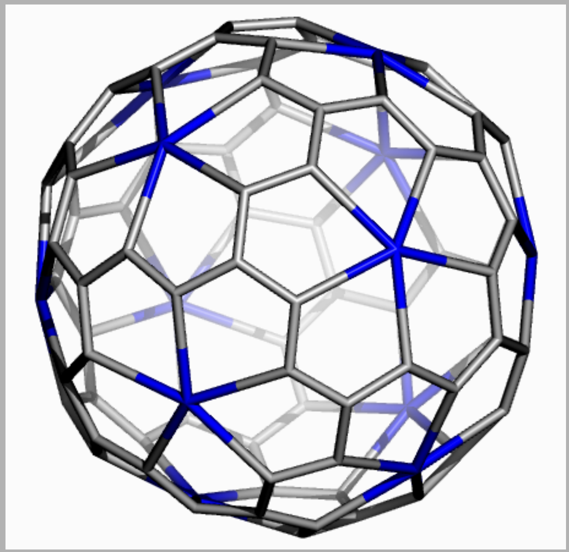


$P_4(Dodecahedron)^1$



1. Catalan objects (*i.e.*, duals of the Archimedean solids).
2. B. de La Vaissière, P. W. Fowler, and M. Deza, *J. Chem. Inf. Comput. Sci.*, **2001**, *41*, 376-386.

P_5 Capping = Pentangulation[1]

$P_5(Cube)$	$P_5(Dodecahedron)$
	

1. For other operation names see www.georgehart.com\virtual-polyhedra \conway_notation.html

Snub *Sn* (*M*)

$$Sn(M) = Dg(Me(Me(M))) = Du(P_5(M))$$

where ***Dg*** is the inscribing diagonals in the tetragons resulted by ***Me*** (***Me*** (***M***)).[1]

The true dual of the snub is the $P_5(M)$ transform.[2]

1. T. Pisanski and M. Randić, in *Geometry at Work*, M. A. A. Notes, **2000**, *53*, 174-194.
2. M. V. Diudea, Forma, 2004 (submitted).

Snub ***Sn*** **(*M*),** continued

Similar to the medial operation,

$$Sn(M) = Sn(Du(M)).$$

In case $M = T$, **the snub** $Sn(M) = I$.

The transformed parameters are:

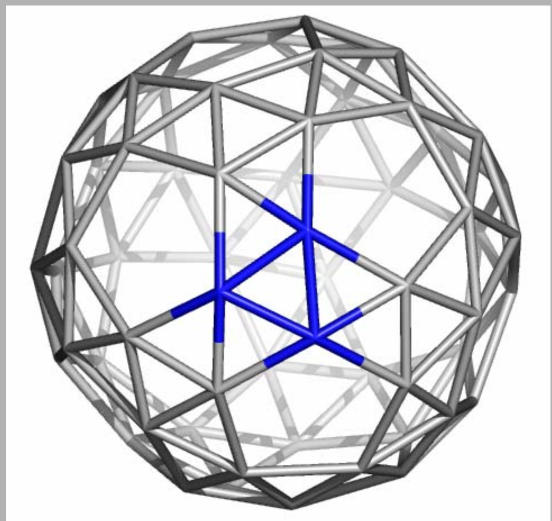
$Sn(M)$:

$$v = s_0 f_0 = d_0 v_0$$
$$e = 5\, e_0$$
$$f = v_0 + 2e_0 + f_0$$

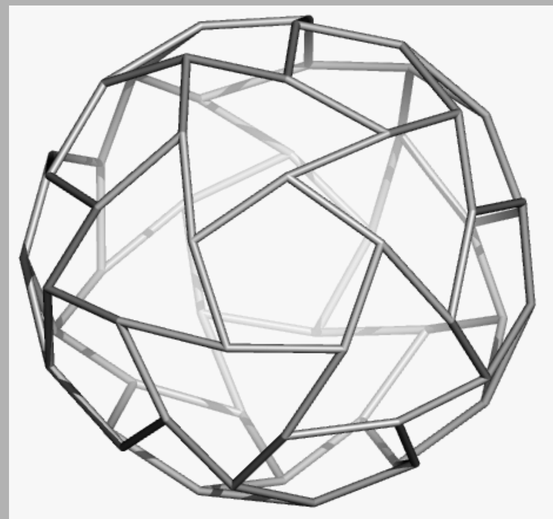
Sn (*M*), examples

Delete the edges of the triangle joining any three parent faces (the blue Triangle – left image) to obtain[2] *Le* (*M*)

Sn (*D*)[1]



Le (*D*)[2] = C_{60}



1. Archimedeans, B. de La Vaissière, P. W. Fowler, and M. Deza, *J. Chem. Inf. Comput. Sci.*, **2001**, *41*, 376-386.
2. M. V. Diudea, Forma, 2004 (submitted).

5. *Leapfrog* [1] = Tripling

- ***Leapfrog Le***

  is a composite operation that can be achieved in two ways:

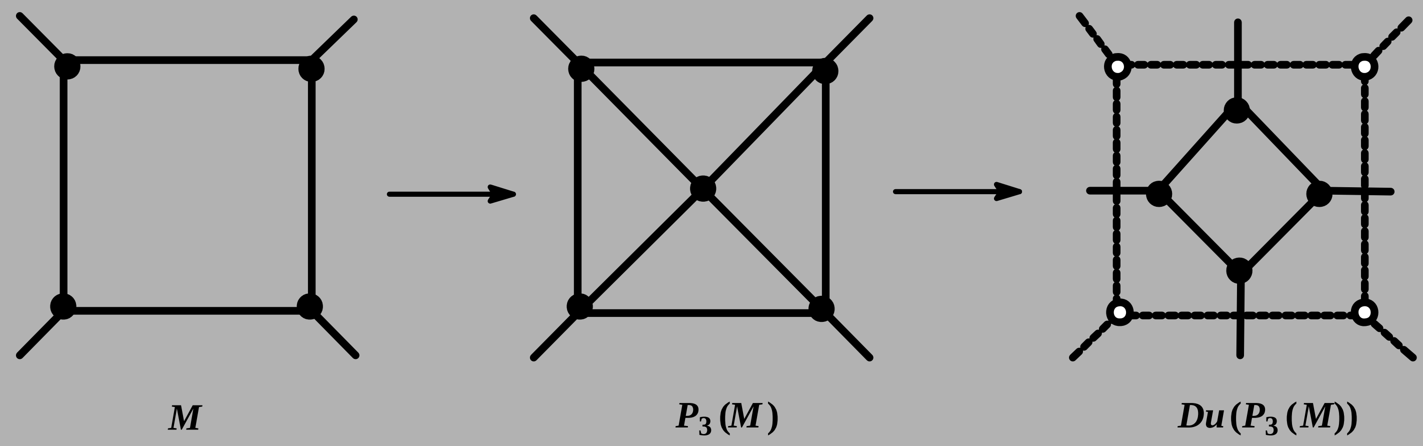
- ***Le*** **(*M*) = *Du*(*P*$_3$(*M*)) = *Tr*(*Du*(*M*))**
- ***Le*** **(*M*)** is always a trivalent graph.

- Within the leapfrog process, the dualization is made on the omnicapped map. *Le* rotates the parent *s*-gonal faces by π/s.

1. P. W. Fowler, How unusual is C_{60}? Magic numbers for carbon clusters. *Chem. Phys. Lett*. 1986, *131*, 444-450.

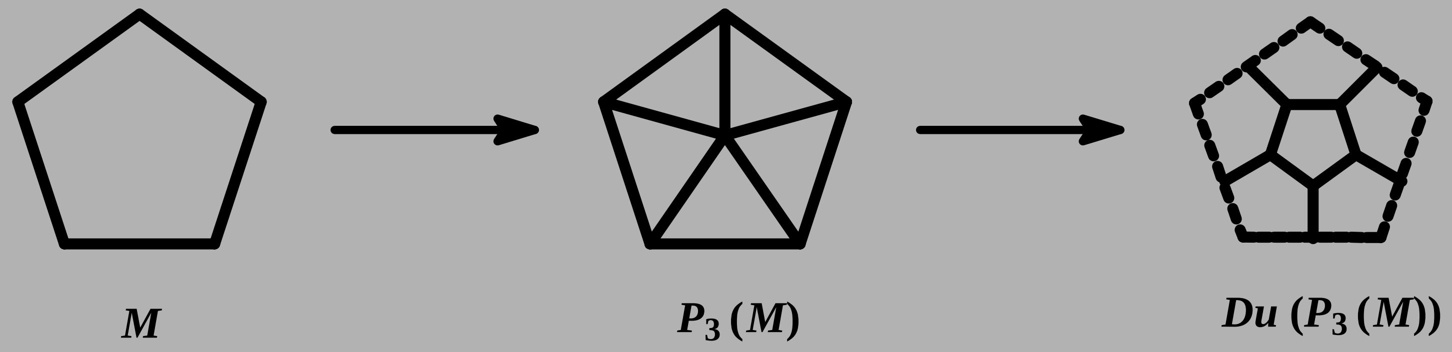
Le , examples

Square face



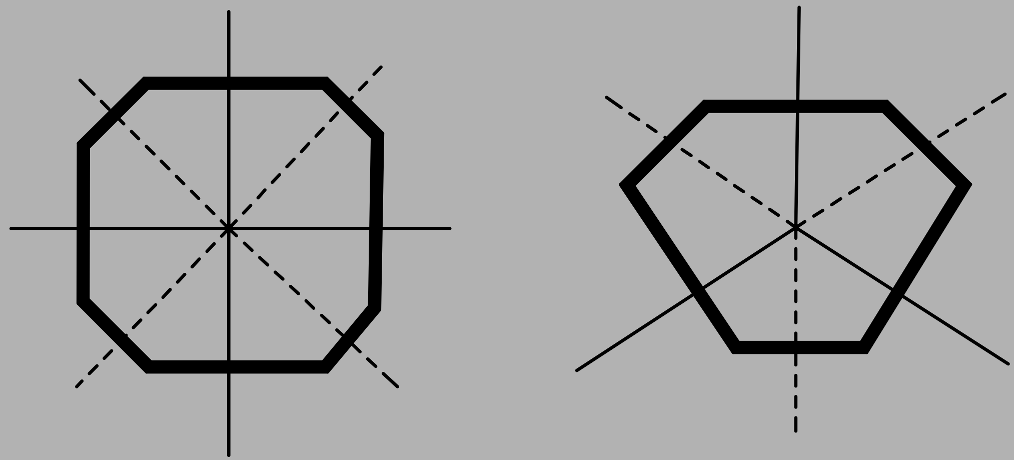
Le , examples

Pentagonal face



Le , examples

Bounding polygon size: $s = 2d_0$



(a) (b)

Theorem on *Le* (*M*)

If *M* is a d_0- regular graph, then:

- *The number of vertices in Le* (*M*) is d_0 *times larger than in the original map M, irrespective of the tessellation type*.

Demonstration: observe that for each vertex v_0 of *M*,

d_0 new vertices result in *Le* (*M*), thus:

$$v/v_0 = d_0 v_0/v_0 = d_0$$

Le (*M*), (continued)

Relations in the transformed map are:

***Le*(*M*):** $v = d_0 v_0 = 2e_0$

$e = 3 e_0$

$f = f_0 + v_0$

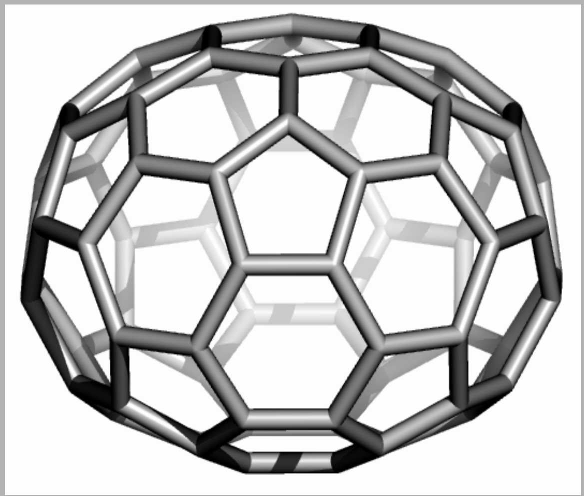
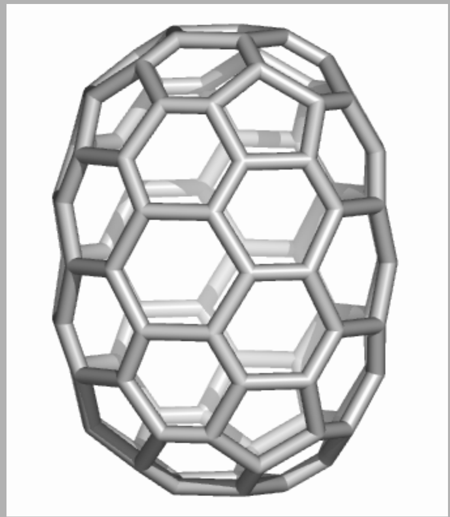
Examples:

Le **(Dodecahedron) = C_{60}**

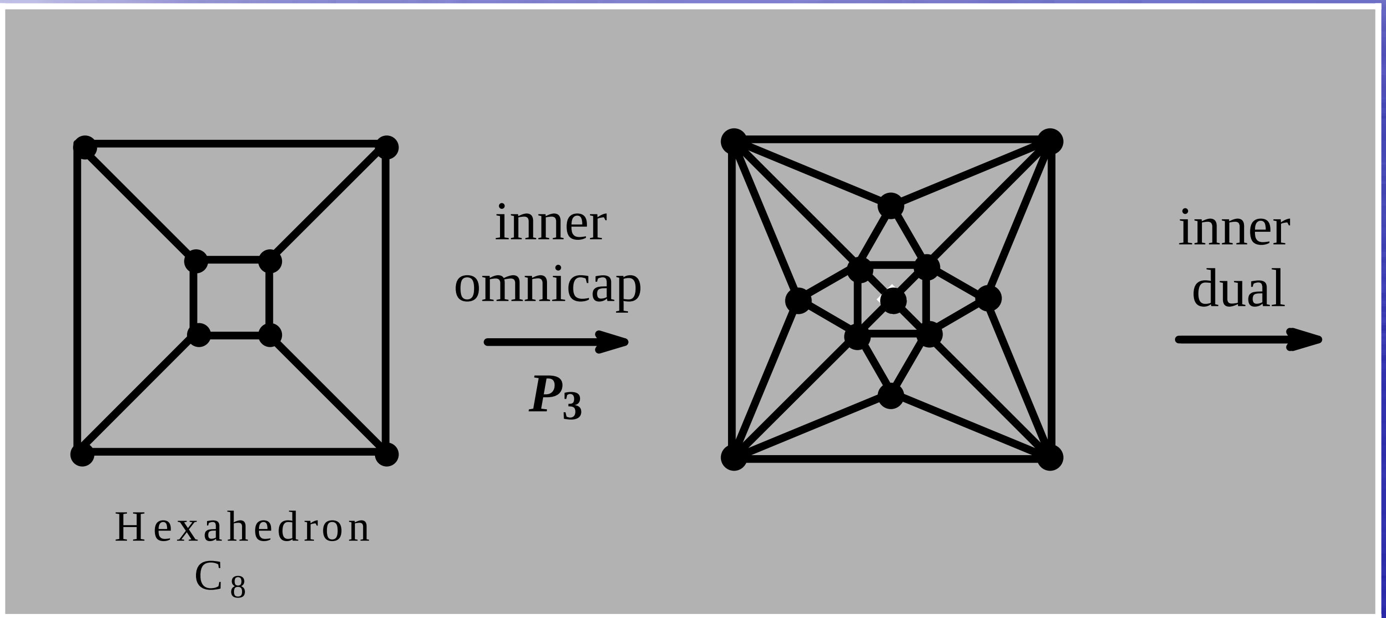
Tr **(Icosahedron) = C_{60}**

Icosahedron = *Du* (Dodecahedron).

***Le* ,** examples

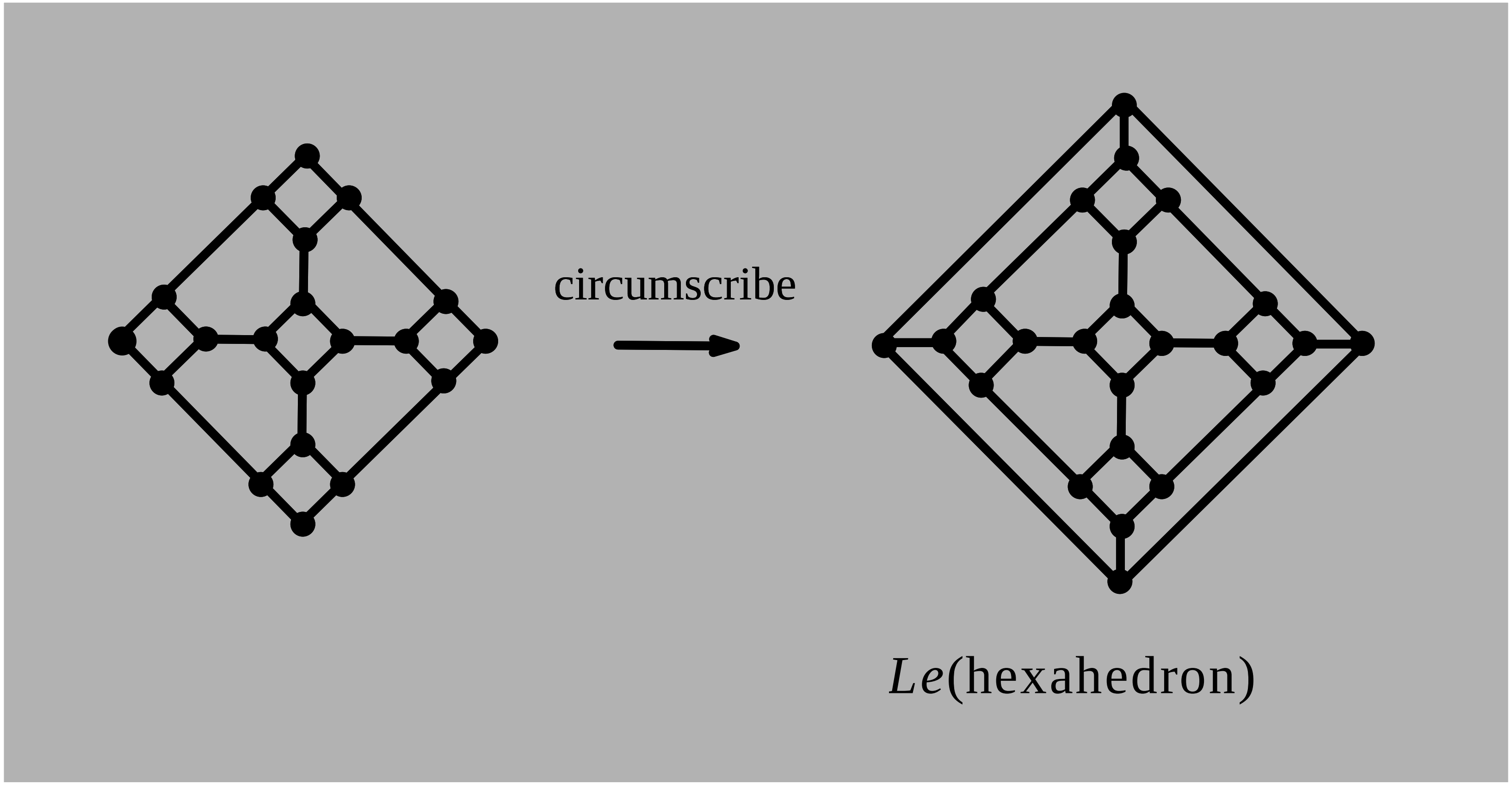
Le (C_{24})	*Le* (C_{30})
	

Schlegel version[1] of *Le* (*M*)

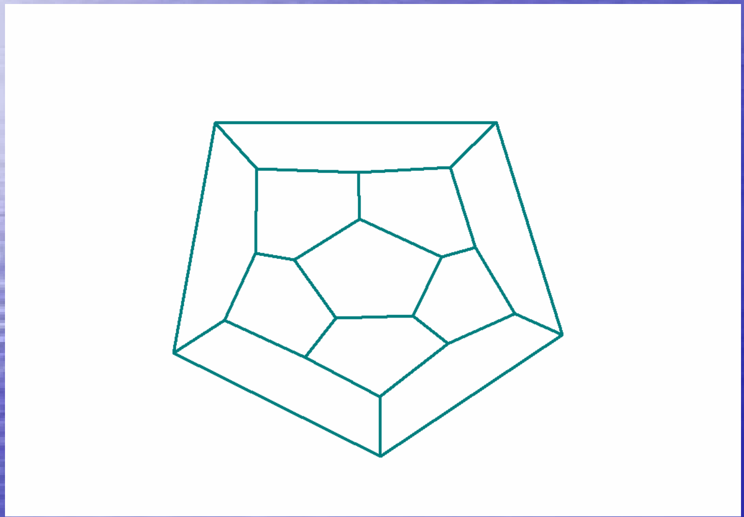
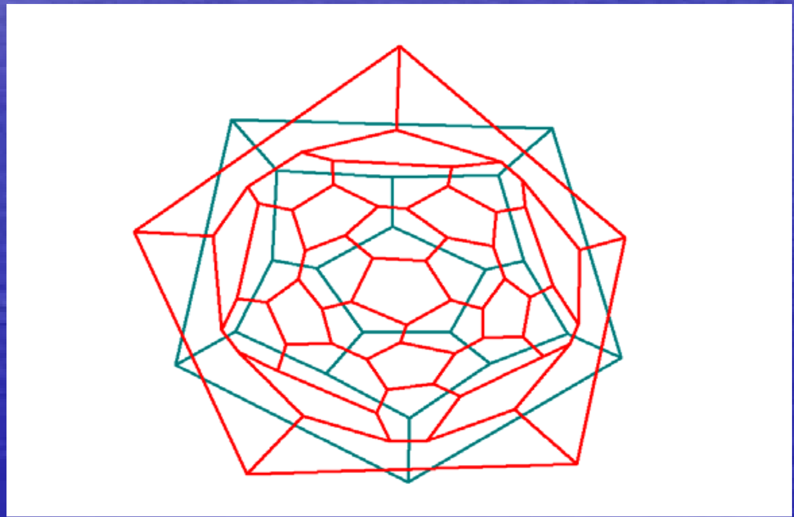


1. J. R. Dias, From benzenoid hydrocarbons to fullerene carbons. *MATCH,Commun. Math. Chem. Comput.* , 1996, *33*, 57-85.

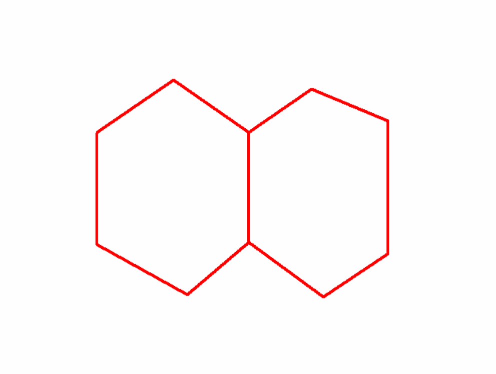
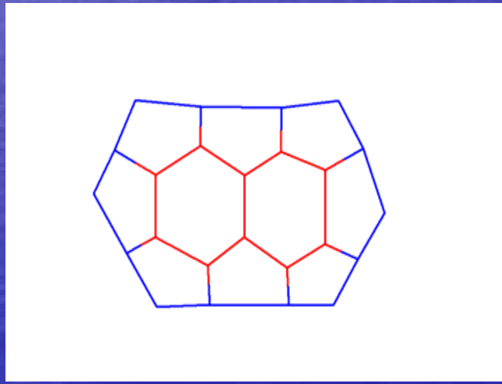
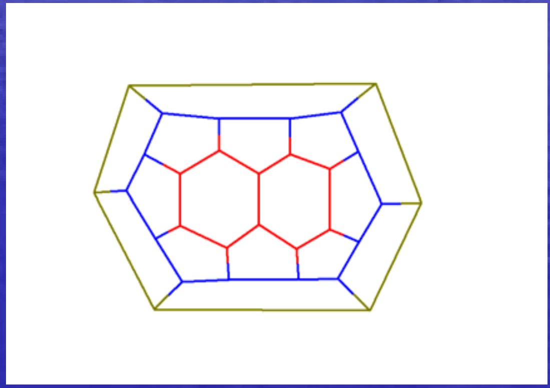
Schlegel version of *Le* (*M*)



Schlegel version of ***Le* (*M*):** examples

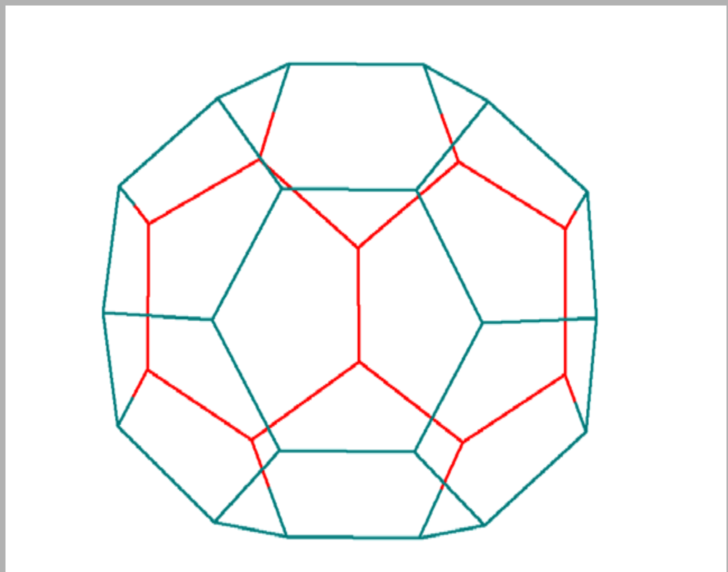
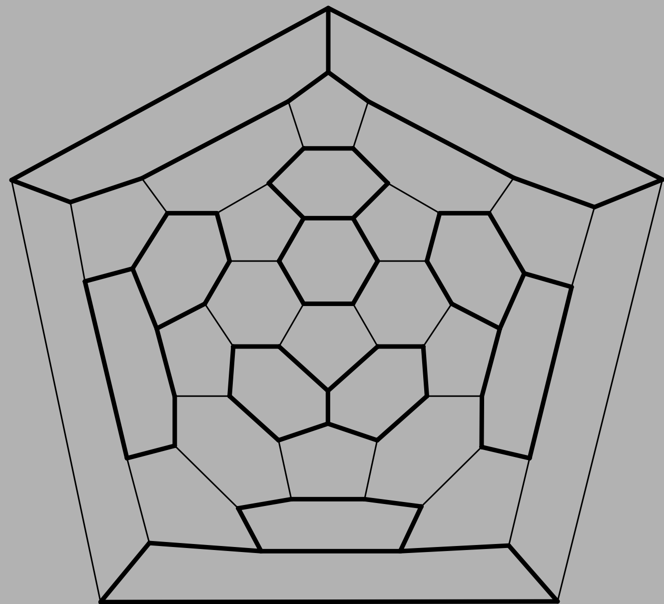
Dodecahedron	*Le* (Dodecahedron)=C_{60}
	

Circumscribing

Naphthalene	5-fold faces	5- and 6-fold faces
		

1. J. R. Dias, From benzenoid hydrocarbons to fullerene carbons. *Commun. Math. Chem. Comput.* (*MATCH*), 1996, *33*, 57-85.

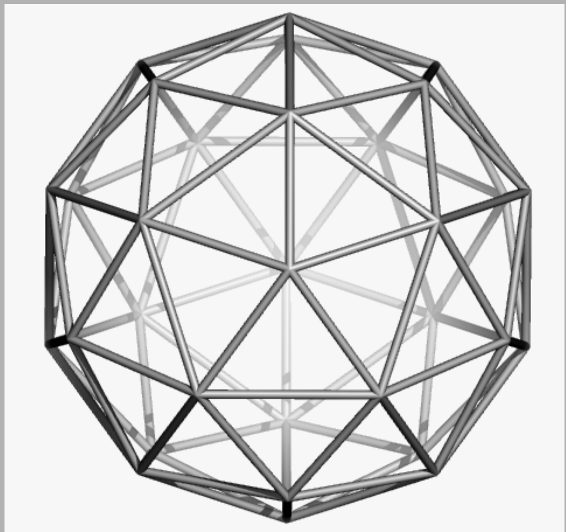
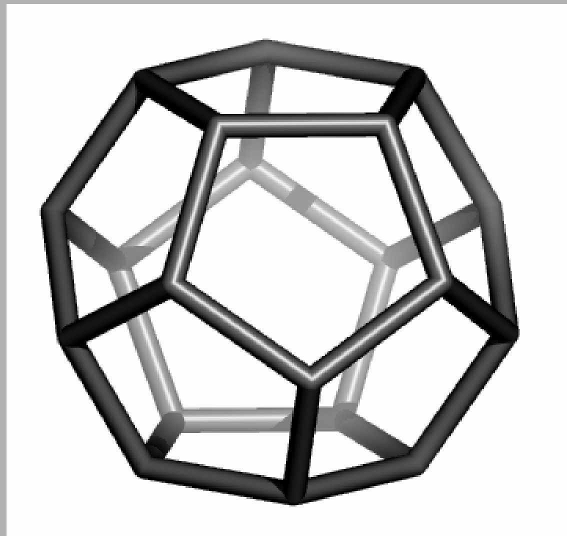
Circumscribing (continued)

C_{30}	Covering C_{60} by 6 naphthalene units
	

Retro-Leapfrog

$$P_3(M_1) = Du(Le(M_1)); \quad M_0 = P_{-3}(M_1)$$

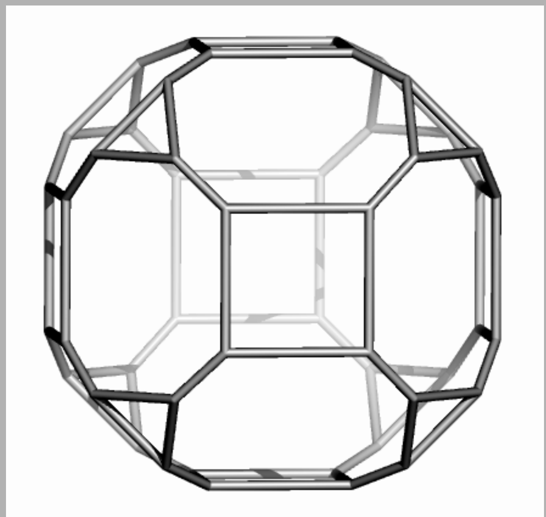
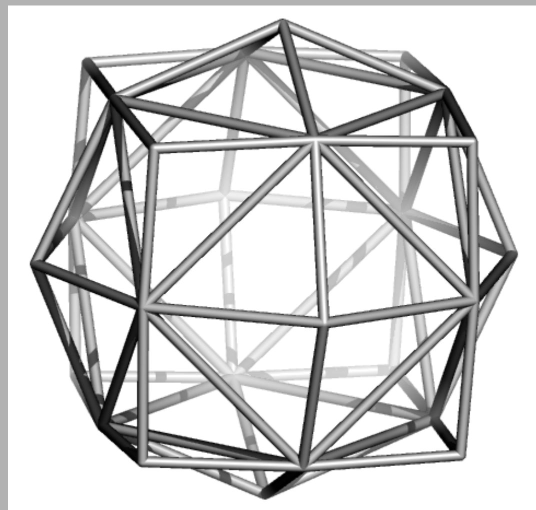
Cut all vertices with degree lower than the maximal one, to get M_0

$Du(Le(C_{20}))$; $N = 32$	D ; $N = 20$
	

Retro-Leapfrog

$$P_3(M_1) = Du(Le(M_1)); \quad M_0 = P_{-3}(M_1)$$

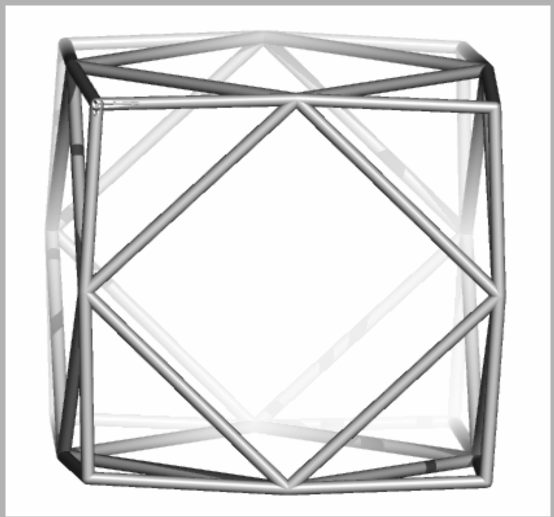
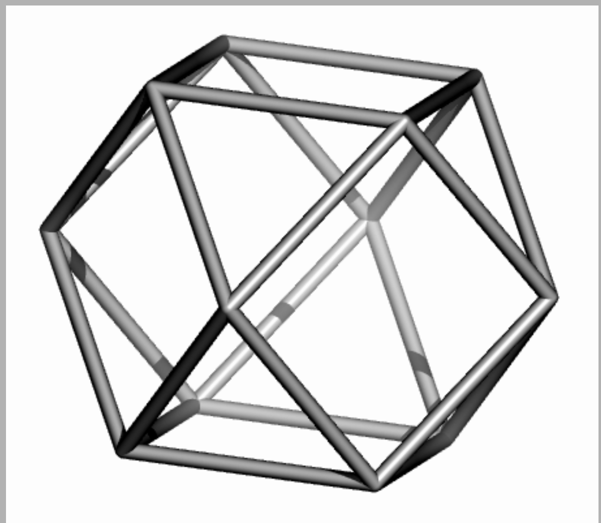
Cut all vertices with degree lower than the maximal one, to get M_0

Le (*Me* (C); *N* = 48	*Du* (*Le* (*Me* (C))
	

Retro-Leapfrog

$$P_3(M_1) = Du(Le(M_1)); \quad M_0 = P_{-3}(M_1)$$

Cut all vertices with degree lower than the maximal one, to get M_0

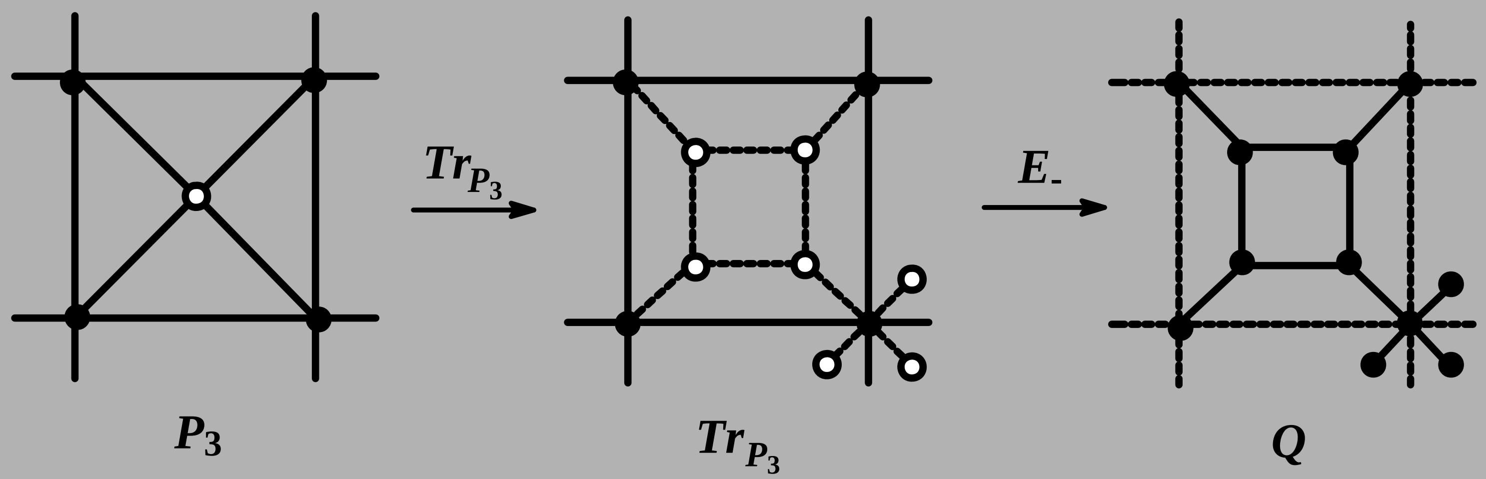
$Du\,(Le\,(Me\,(\mathrm{C})))$_$XC_4$ $N = 20$	Cubeoctahedron = $Me\,(C)$; $N = 12$
	

6. *Quadrupling* $Q(M)$

- **Chamfering = Edge Truncation**
- $Q(M) = E_{-}(Tr_{P3}(P_3(M)))$
- Q **operation leaves unchanged the initial orientation of the polygonal faces.**

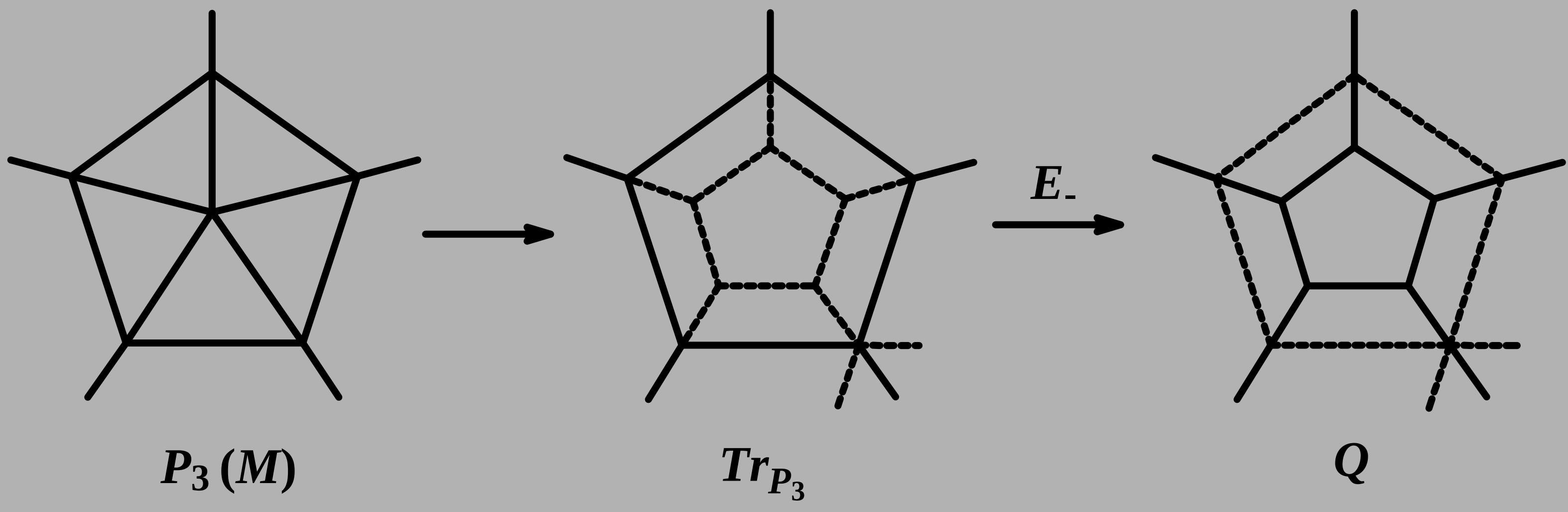
$Q(M)$, examples

Square face



$Q(M) = E_{-}(Tr_{P_3}(P_3(M)))$, examples

Pentagonal face



Theorem on $Q(M)$

- *The vertex multiplication ratio in $Q(M)$ is $d_0 + 1$ irrespective of the tiling type of M.*
- Demonstration: observe that for each vertex v_0 in M results d_0 new vertices in $Q(M)$ and the old vertices are preserved.

Thus: $v = d_0 v_0 + v_0$

$$v / v_0 = (d_0+1) v_0 / v_0 = d_0+1$$

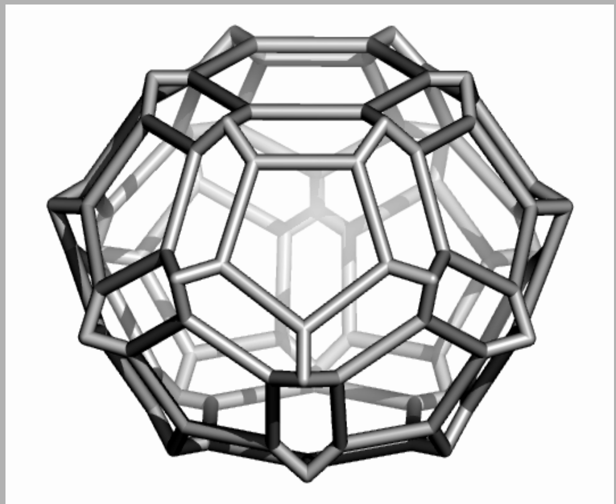
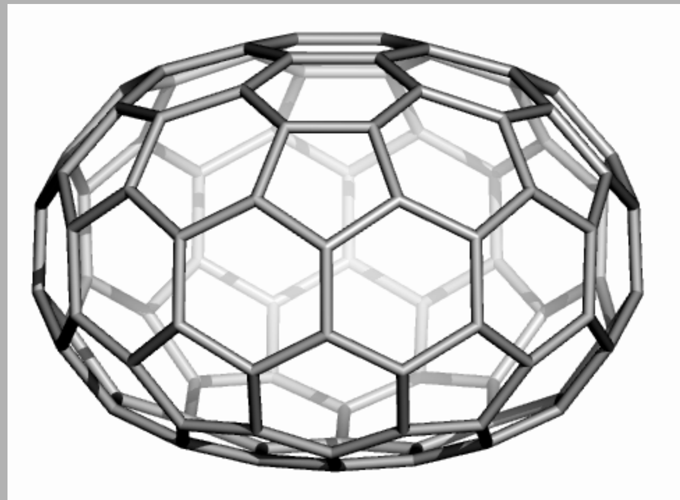
Q (*M*) (continued)

- The transformed parameters are:

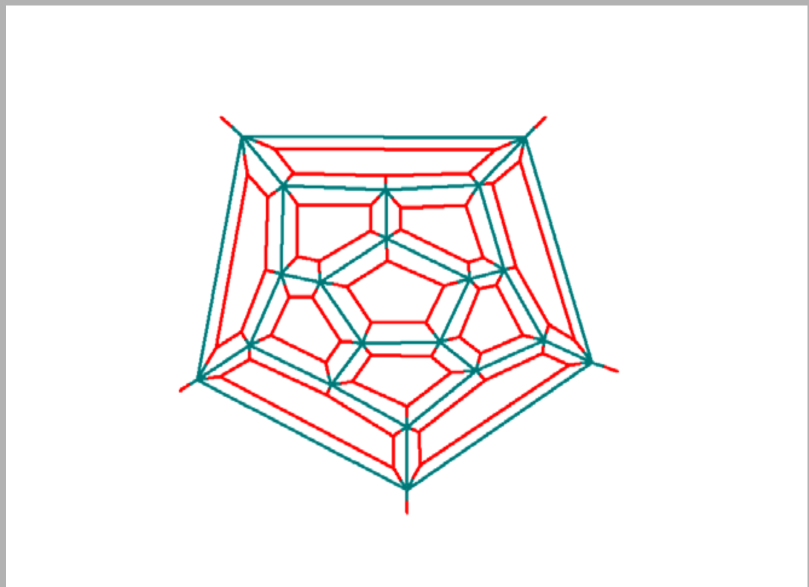
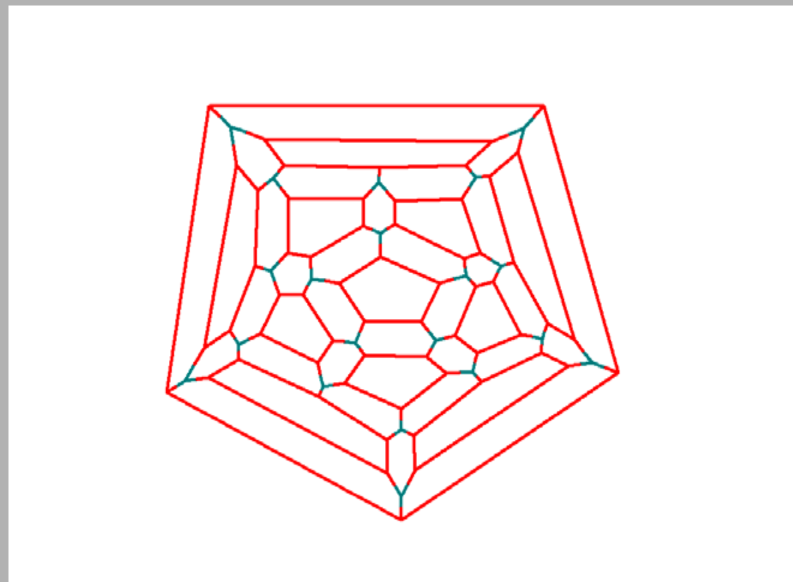
- *Q* (*M*): $v = (d_0+1)v_0$
  $e = 4e_0$
  $f = f_0+e_0$

- In a 4-valent map, *Q* leads to a non-regular graph (3- and 4-valent vertices).

Q (*M*), examples

$Q\,(C_{24})$ (non-optimized)	$Q\,(C_{24})$ (optimized)
	

Schlegel version of *Q* (*M*)

Q (Dodecahedron)	C_{80u}
	

Platonic and Archimedean polyhedra (derived from Tetrahedron)

	Symbol	Polyhedron	Formula
1	T	Tetrahedron	-
2	O	Octahedron	$Me(T)$
3	C	Cube (hexahedron)	$Du(O)=Du(Me(T))$
4	I	Icosahedron	$Sn(T)$
5	D	Dodecahedron	$Du(I) = Du(Sn(T))$
1	TT	Truncated tetrahedron	$Tr(T)$
2	TO	Truncated octahedron	$Tr(O) = Tr(Me(T))$
3	TC	Truncated cube	$Tr(C) = Tr(Du(Me(T)))$
4	TI	Truncated icosahedron	$Tr(I) = Tr(Sn(T))$
5	TD	Truncated dodecahedron	$Tr(D) = Tr(Du(Sn(T)))$
6	CO	Cuboctahedron	$Me(C) = Me(O)= Me(Me(T))$
7	ID	Icosidodecahedron	$Me(I) = Me(D) = Me(Sn(T))$
8	RCO	Rhombicuboctahedron	$Me(CO) = \mathbf{Me(Me(C)) = Du(P_4(C))}$
9	RID	Rhombicosidodecahedron	$Me(ID) = \mathbf{Me(Me(I)) = Du(P_4(I))}$
10	TCO	Truncated cuboctahedron	$Tr(CO) = Tr(Me(Me(T)))$
11	TID	Truncated icosidodecahedron	$Tr(ID) = Tr(Me(Sn(T)))$
12	SC	Snub cube	$Sn(C) = Du(P_5(C)) = Du(Op(Ca(C)))$
13	SD	Snub dodecahedron	$Sn(D) = Du(P_5(D)) = Du(Op(Ca(D)))$